Delta hedging with volatility skew

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Abstract:
Practitioners often price options based on the Black-Scholes Model. However, the constant volatility assumption in the model can be easily violated in the real financial market. In practice, volatility can vary with time, underlying asset price or strike price. Since volatility is an important part in the calculation of delta, incorrect volatility will lead to inaccurate delta. Various adjustments on delta have been proposed by previous research papers. This research will focus on the fact of volatility skew. Equipped with the functional programming language, Clojure, the research will test the improvement of delta using the Clojure back-testing library. It will compare reduction in variance to generate conclusion.

Theoretical Model

In the Black Scholes model, the put option delta is calculated as follows,

$$\Delta_{BS} = \frac{\partial P(S,K,\sigma,r,T)}{\partial S} = -N\left(-\frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

The constant volatility assumption in the Black Scholes model is always violated in the real financial market. Volatility actually varies across time and is negatively correlated with the underlying stock price most of the time. Hence, an adjusted delta is generated by taking into account the volatility changes related to the underlying price change $\frac{\partial\sigma}{\partial S}$. By the chain rule, the adjusted delta is expressed as

$$\delta_{adj} = \frac{\partial P}{\partial S} + \frac{\partial P}{\partial \sigma}\frac{\partial \sigma}{\partial S}$$

The first multiplier is the Vega of the put option by definition, whereas the second term is usually difficult to estimate. As stated by Coleman et al. (2001), the second multiplier can be estimated by the slope of the volatility smile $\frac{\partial\sigma}{\partial K}$. To simplify the model, the slope is estimated by a linear relationship of the implied volatility and the strike price of out-of-the-money strike price.

Data and Strategy

Both the stock data and the option data span for three years from 2016 to 2018. The stock data contains the necessary information such as date, close and opening price, daily return, and so on. The option data only includes the put option information that will be used for the trade back-testing. This was done by manually selecting the required information and specifying the time to expiration.

The main idea of the strategy is to make sure the combined position remains delta-neutral by adjusting the portfolio daily. It starts from holding a fixed amount of underlying assets. In particular, the AAPL stock is traded in the strategy. Every day, five put option contracts are selected. Their combined delta is calculated as the weighted average of the individual delta.

It follows that the subsequently decided based on the delta-neutral rule. Every day, the strategy sells the contracts that are no longer in the portfolio on the bid price and purchases the contracts that are newly added on the asking price. Finally, the total portfolio value is generated by adding the value of all the assets in the portfolio together.

In the evaluation spreadsheet, the return rate is calculated daily according to the simple return calculation. The BS-delta and adjusted delta will have different return rates. We compare their average return. More importantly, comparing the standard deviation of returns in the two strategies will give information on stability.

Result summary

Table 1 gives information on the summary statistics of how the strategies work in the three years. We can observe that the average return of the Black Scholes model is always higher than the adjusted volatility model across the three years. However, the standard deviation in the Black Scholes model is higher than the adjusted model. In the calculation Sharpe ratio, the situation is simplified by assuming the average risk-free rate is 0.1%. Because both average return and standard deviation in the adjusted model are less than those in the original BS model, the scales of the Sharpe ratio vary. In terms of the Sharpe ratio, the BS model does better in the year 2016 and 2017. On the other hand, the adjusted model in 2018 outperforms a lot in 2018 because of the substantial reduction in variation.

Regarding the main purpose of this research, we focus on the value of standard deviation because it is the measure of the stability of the return. Therefore, we may conclude that by taking the slope of the volatility skew into account and using a corresponding adjusted delta for hedging can improve the stability of the whole position.

Conclusion

With the existence of volatility skew, the Greeks calculated based on the Black Scholes model is regarded inaccurate. This is because of the constant volatility assumption does not happen in the real financial market. The research specifically focuses on the negative correlation between the volatility and the underlying asset price.

By hedging based on an adjusted delta, the overall stability of the delta neutral strategy improves. This indicates that adjustments should be made according to the real market condition and investors' view of the asset volatility level. Future studies may also take other market facts into account and also the real market frictions. This will make the strategy less exposed to other risks and possible to generate more profit.

References:

Nian, K., Coleman, T., & Li, Y. (2018). Learning minimum variance discrete hedging directly from the market. Quantitative Finance, 18(7), 1115-1128.

Vähämaa, S. (2004). Delta hedging with the smile. Finanzmarkt Und Portfolio Management, 18(3), 241-255.

	2016		**2017**		**2018**	
	BS-model	Adjusted	BS-model	Adjusted	BS-model	Adjusted
average return	0.2939%	0.2737%	0.3543%	0.3144%	0.7753%	0.6826%
standard deviation	0.4250%	0.3971%	0.5944%	0.5089%	1.1791%	0.7148%
Sharpe ratio	0.45621	0.43754	0.42790	0.42128	0.57274	0.81505

Table1: summary statistics