Research report: delta hedging with volatility skew

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Introduction

In the options market, delta is a measure of exposure of option price to the price of the underlying asset. The deltas posted on the trading board or information website are mostly calculated based on the Black Scholes model. The fundamental assumption in the model is constant volatility. This does not happen in the real financial market, which will make the calculation of option Greeks inaccurate. Hence influencing the stability of delta hedged potions. This research aims to adjust the calculation of delta by considering volatility-skew. In particular, the negative correlation between the volatility and the strike price. The research will test whether the adjusted delta does better in stabilizing a delta neutral position than the traditional Black Scholes (BS) delta.

As one of the main motivations for this project, all the back-testing processes are done using the Clojure back-testing library.

For the content below, the report is organized as follows. It will first introduce the theoretical model and calculation of the strategy. After that, it will show the data used in the research and the trading rules in the strategy. Then, the report will demonstrate the results generated using the Clojure back-testing library. In the following part, the paper will also point out the limitations of the research. The conclusion part will summarize the whole paper at the end.

Theoretical Model

In the Black Scholes model, the put option delta is calculated as follows,

$$\Delta_{BS} = \frac{\partial P(S,K,\sigma,r,T)}{\partial S}$$
$$= -N\left(-\frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

The constant volatility assumption in the Black Scholes model is always violated in the real financial market. Volatility actually varies across time and is negatively correlated with the underlying stock price most of the time. Hence, an adjusted delta is generated by taking into account the volatility changes related to the underlying price change $\frac{\partial\sigma}{\partial S}$. By the chain rule, the adjusted delta is expressed as

$$\delta_{adj} = \frac{\partial P}{\partial S} + \frac{\partial P}{\partial \sigma}\frac{\partial \sigma}{\partial S}$$

The first multiplier is the Vega of the put option by definition, whereas the second term is usually difficult to estimate. As stated by Nian et al. (2018), the second multiplier can be estimated by the slope of the volatility smile $\frac{\partial\sigma}{\partial K}$. As observed in the scatter plot, the slope of the volatility simile is negative for out of the money puts and positive for in the money puts. To simplify the model, the slope is estimated by a linear relationship of the implied volatility and the strike price of out-of-the-money strike price.

Data and Strategy

The research strategy is based on the underlying asset and its corresponding put options. All the assets information was downloaded from the WRDS website.

Both the stock data and the option data span for three years from 2016 to 2018. The stock data contains the necessary information such as date,

	2016		2017		2018	
	BS-model	Adjusted	BS-model	Adjusted	BS-model	Adjusted
average return	0.2939%	0.2737%	0.3543%	0.3144%	0.7753%	0.6826%
standard deviation	0.4250%	0.3971%	0.5944%	0.5089%	1.1791%	0.7148%
Sharpe ratio	0.45621	0.43754	0.42790	0.42128	0.57274	0.81505

Table1: summary statistics

close and opening price, daily return, and so on. The option data only includes the put option information that will be used for the trade back-testing. This was done by manually selecting the required information and specifying the time to expiration.

The main idea of the strategy is to make sure the combined position remains delta-neutral by adjusting the portfolio daily. It starts from holding a fixed amount of underlying assets. In particular, the AAPL stock is traded in the strategy. Every day, five put option contracts are selected. They are selected as the top five most traded options during the day. Hence, they may have the most liquidity. Their combined delta is calculated as the weighted average of the individual delta. It follows that the subsequently decided based on the delta-neutral rule. Every day, the strategy sells the contracts that are no longer in the portfolio on the bid price and purchases the contracts that are newly added on the asking price. Finally, the total portfolio value is generated by adding the value of all the assets in the portfolio together.

In the evaluation spreadsheet, the return rate is calculated daily according to the simple return calculation.

$$\frac{V_t}{V_{t-1}} - 1$$

The BS-delta and adjusted delta will have different return rates. We compare their average return. More importantly, comparing the standard deviation of returns in the two strategies will give information on stability.

Result summary and visualization

Table 1 gives information on the summary statistics of how the strategies work in the three years.

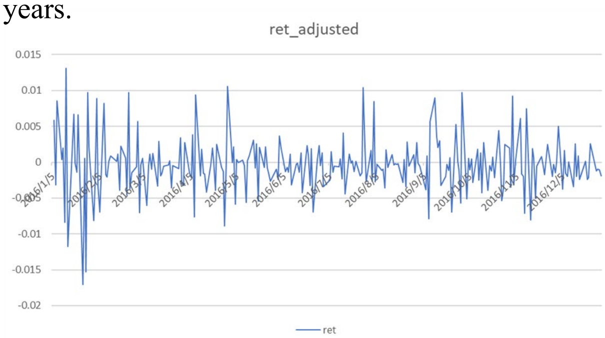

Figure 1: return for 2016

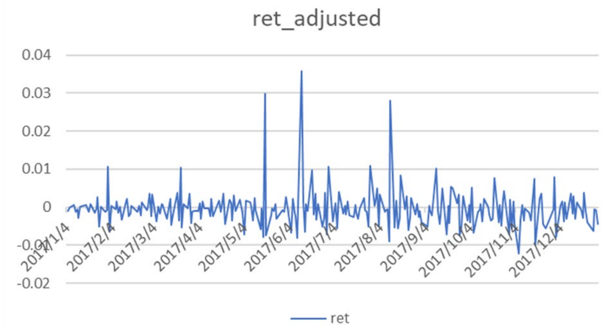

Figure 2: return for 2017

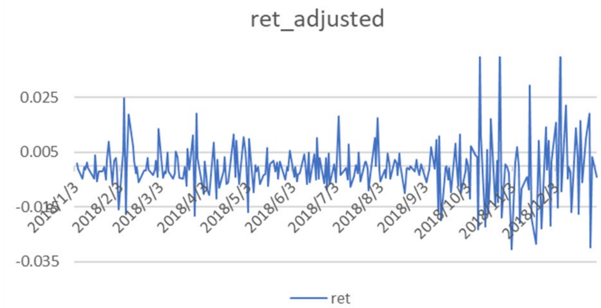

Figure 3: return for 2018

As a reference, Figures 1 to 3 show the daily return using the adjusted delta for hedging.

We can observe that the average return of the Black Scholes model is always higher than the adjusted volatility model across the three years. However, the standard deviation in the Black Scholes model is higher than the adjusted model. In the calculation Sharpe ratio, the situation is simplified by assuming the average risk-free rate is 0.1%. Because both average return and standard deviation in the adjusted model are less than those in the original BS model, the scales of the Sharpe ratio vary. In terms of the Sharpe ratio, the BS model does better in the year 2016 and 2017. On the other hand, the adjusted model in 2018 outperforms a lot in 2018 because of the substantial reduction in variation.

Regarding the main purpose of this research, we focus on the value of standard deviation because it is the measure of the stability of the return. Therefore, we may conclude that by taking the slope of the volatility skew into account and using a corresponding adjusted delta for hedging can improve the stability of the whole position.

Limitations

As a native trading strategy, it has some apparent limitations that should be considered if real trade is executed.

First, the option ordering function is user-coded and did not include the transaction cost and interest rate parameters. This will not happen in the real market. We need to note that the portfolio is adjusted daily, and thus will incur an amount of transaction cost that should not be ignored in the real strategy building. To solve this issue, the trader could find a balanced adjustment frequency, such as every five or three days to lower the transaction cost but still maintain delta neutrality.

Second, the reader should note that as a strategy used for research, the main idea is not to maximize the profit, but to if the new delta does better in stabilizing the delta neutral position. Therefore, there are other risk exposures left except that related to pricing change of the underlying, such as Vega risk or second derivatives of gamma risk and so on. Hence, in the real market, where profiting is the initial goal, there should be more complex trading of various instruments.

Finally, by using the trading data, the research is implicitly assuming that all the orders are executed without rejection from the trading market. This will also not happen in the real financial market because market friction always exists. Hence, there should be different degrees of influence on both models, making the statistics not quite accurate.

Conclusion

With the existence of volatility skew, the Greeks calculated based on the Black Scholes model is regarded inaccurate. This is because of the constant volatility assumption does not happen in the real financial market. The research specifically takes the negative correlation between the volatility and the underlying asset price into consideration.

By hedging based on an adjusted delta, the overall stability of the delta neutral strategy improves. This indicates that adjustments should be made according to the real market condition and investors' view of the asset volatility level. Future studies may also take other market facts into account and also the real market frictions. This will make the strategy less exposed to other risks and possible to generate more profit.

References:

Nian, K., Coleman, T., & Li, Y. (2018). Learning minimum variance discrete hedging directly from the market. Quantitative Finance, 18(7), 1115-1128.

Vähämaa, S. (2004). Delta hedging with the smile. Finanzmarkt Und Portfolio Management, 18(3), 241-255.